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THE CONSTRUCTION OF A SET OF RECURRENCE WHICH IS NOT A SET OF STRONG RECURRENCE*

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ABSTRACT

This paper deals with two possible definitions of recurrence in measure preserving systems. A set of integers R is said to be a set of (Poincaré) recurrence if, for all measure preserving systems $(X, \mathcal{B}, \mu, T)$ and any measurable set A of positive measure, there is an $r \in R$ such that $\mu(T^r A \cap A) > 0$. R is said to be a set of strong recurrence if, for all measure preserving systems $(X, \mathcal{B}, \mu, T)$ and any measurable set A of positive measure, there is an $e > 0$ and an infinite number of elements r of R such that $\mu(T^r A \cap A) \geq e$ (see Bergelson's 1985 paper). This paper constructs a set of recurrence R, an example of a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a measurable set A of measure $1/2$, such that $\lim_{r \to \infty : r \in R} \mu(A \cap T^r A) = 0$. In particular R is a set of recurrence but not a set of strong recurrence, giving a negative answer to a question of Bergelson posed in 1985. Further, it also constructs a set of recurrence which does not force the continuity of positive measures and so reproves a result of Bourgain published in 1987.

1. The construction

This first makes an approximation in a finite product of Z_2 ($= \{0, 1\}$, addition mod 2), then puts better and better approximations together in Z_2^∞ to obtain an appropriate counterexample for Z_2^∞ actions. This is then transferred, by combinatorial methods, to $\mathbb{Z}$.

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Definitions: The following definitions will be used throughout the paper.

Let G be an abelian group; usually $\mathbb{Z}, Z_2^\infty$, or Z_P, in what follows.

A **G action** on a probability measure space, $(X, \mathcal{B}, \mu)$, is a representation of G by measure preserving maps: $T_g : X \to X$ defined for each $g \in G$, i.e.,

$$T_g T_h = T_{g+h} \text{ and } \mu(A) = \mu(T_g A), \text{ for all } g \text{ and } h \in G, \text{ and } A \in \mathcal{B}.$$

This is written $(X, \mathcal{B}, \mu, G)$ or $(X, \mathcal{B}, \mu, T)$ and, where G is understood, is called simply a **measure preserving system.** Often T_g will be written T^g.

Define, for any subset R of G, the function $e : [0,1] \to [0,1]$,

$$e(a; R) = \inf_{(X,\mathcal{B},\mu,T);\mu(A)\geq a} \quad \sup_{r\in R} \mu(A \cap T_r A),$$

where the infimum is taken over all probability measure preserving dynamical systems $(X, \mathcal{B}, \mu, T)$ upon which the group acts, and sets A of measure at least a.

Let $L(R) = \inf\{a : e(a; R) > 0\}$.

A subset R of G is a **set of** (Poincaré) **recurrence** if, for all measure preserving G actions $(X, \mathcal{B}, \mu, T)$ and any measurable set A of positive measure, there is an $r \in R$ such that $\mu(T^r A \cap A) > 0$.

R is said to be a **set of strong recurrence** if, for all measure preserving G actions $(X, \mathcal{B}\mu, T)$ and any measurable set A of positive measure, there is an $e > 0$ and an infinite number of elements r of R such that $\mu(T^r A \cap A) \geq e$ (see [1]). ∎

Remark: Poincaré recurrence may be shown to be equivalent to the requirement that $e(a; R)$ be strictly positive on $(0,1]$ as a function of a, i.e. $L(R) = 0$. In addition, it may be shown that

$$L(R) = \sup\{a : \exists (X, \mathcal{B}, \mu, G) \text{ and } A : \mu(A) = a : \mu(A \cap T^r A) = 0 \text{ for all } r \in R\}.$$

∎

THEOREM 1.1: *There is a set of recurrence $R \in \mathbb{Z}$, a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set A of measure 1/2, for which*

$$\lim_{r\to\infty : r\in R} \mu(A \cap T^r A) = 0.$$

Proof: See section 3.

A set S forces the continuity of positive measures (is FC^+) if every positive measure m on the unit circle which has the property:

$$\lim_{s\to\infty: s\in S} m\hat{}(s) = 0$$

is necessarily continuous (i.e. is free of atoms).

Bourgain [3], on the way to proving a much stronger result, constructs a set of recurrence which is not FC^+. It is now possible to construct this in a different way.

THEOREM 1.2 (Bourgain [3]): *There is a set of recurrence which is not FC^+.*

Proof: Use the construction of Theorem 1.1 directly.

Set m to be the positive real measure on the unit circle whose Fourier transform is the positive definite sequence $m\hat{}(n) = \mu(T^n A \cap A)$.

m has an atom since $\mu(A) > 0$ and so the continuity of m is not forced, although $m\hat{}(n)$ tends to zero along R.

However R is a set of recurrence. ∎

The following lemma allows L to be calculated quite easily in the case of finite groups.

LEMMA 1.3: *Let G be a finite abelian group and let R be a subset. Define $L\hat{}(R) = \max\{|A|/|G| : G \supseteq A$ and $A \cap A + r = \emptyset$ for all $r \in R\}$.*

Then $L\hat{}(R) = L(R)$.

Proof: That $L \geq L\hat{}$ is obvious since G acts on itself in a measure preserving manner, the measure in question being the normalized counting measure. On the other hand suppose that $(X, \mathcal{B}, \mu, G)$ is a measure preserving system, and let A be a subset of measure $a > L\hat{}(R)$.

Therefore there is a subset E of G of cardinality greater than $a \cdot |G|$ for which $\mu(\cap_{g\in E} T_g A) > 0$. By the definition of $L\hat{}$, there is an element r of R for which $E \cap E + r \neq \emptyset$. This implies that $\mu(A \cap T^r A) > 0$ and we are done. ∎

2. The Result in Z_2^∞

Here Z_2^∞ is considered to be the collection of all finitely supported functions from $\mathbb{N}$ to $\{0, 1\}$ and functional notation is used.

Let E be a subset of $\mathbb{N}$. Define $Z_2^E = \{v \in Z_2^\infty : v(n) = 0 \text{ if } n \notin E\}$. Without confusion, the set $\{a, a+1, \ldots, b\}$ will be denoted $[a, b]$, the set $\{a, a+1, \ldots, b-1\}$ will be denoted $[a, b)$, etc.

The upper density of a subset U of Z_2^∞ is defined as

$$\bar{d}(U) = \overline{\lim_{n\to\infty}} \frac{|U \cap Z_2^{[1,n]}|}{2^n}.$$

THEOREM 2.1: *There is a set of recurrence R and a measure preserving system $(X, \mathcal{B}, \mu, Z_2^\infty)$ with a set A, of measure equal to 1/2, for which $\mu(T_r A \cap A)$ tends to zero as r tends to infinity along R (i.e. for all $\varepsilon > 0$ there is an N such that, if $\mu(T_r A \cap A) > \varepsilon$ and r is in R, then r is in $Z_2^{[1,N)}$).*

This theorem clearly produces a set of recurrence which is not a set of strong recurrence in Z_2^∞.

Note that it is sufficient to find a subset U of Z_2^∞ and a sequence of integers n_k tending to infinity for which

$$\lim_{k\to\infty} \frac{|U \cap Z_2^{[1,n_k]}|}{2^{n_k}} = \frac{1}{2} \text{ and } \lim_{r\to\infty : r\in R} \lim_{k\to\infty} \frac{|U \cap (U+r) \cap Z_2^{[1,n_k]}|}{2^{n_k}} = 0.$$

See Bergelson [1] for the equivalence of recurrence and density intersectivity in the case of a general amenable group.

Consider the product Z_2^{2N}, where N is a large integer.

An element $\mathbf{a} = (a_1, a_2, \ldots, a_{2N}) \in Z_2^{2N}$ has various interpretations which will appear in the work which follows:

First, $\mathbf{a}$ may be considered as a vector in $\mathbb{R}^{2N}$ and, as such, has an l_1 norm, $|\mathbf{a}|_1 = \Sigma_{i=1}^{2N} a_i$, the sum here being taken in $\mathbb{R}$ and not Z_2. This imposes a metric on Z_2^{2N}, namely $d(\mathbf{a}, \mathbf{b}) = |\mathbf{a} - \mathbf{b}|_1 = |\mathbf{a} + \mathbf{b}|_1$.

Alternatively, $\mathbf{a}$ may be thought of as the indicator of a subset A of $\{1, \ldots, 2N\} : A = \{i : a_i = 1\}$. Clearly, $\text{card}(A \Delta B) = d(\mathbf{a}, \mathbf{b})$.

Given $M < N$, define $R = R(N, M) = \{\mathbf{r} \in Z_2^{2N} : |\mathbf{r}|_1 > 2M\}$.

Suppose that V is a subset of Z_2^{2N} for which V and $V + \mathbf{r}$ are disjoint for all choices of $\mathbf{r} \in R$. Thus for all $\mathbf{v}$ and $\mathbf{v}'$ in V, $|\mathbf{v} - \mathbf{v}'|_1 \le 2M$; in other words, the diameter of V is at most $2M$. A theorem of Kleitman [5] then says that V can have at most $\Sigma_{i\le M} \binom{2N}{i}$ elements.

By the normal approximation of the binomial distribution, this sum is asymptotically equal to $2^{2N} \cdot \Phi\left((M - N)/\sqrt{N/2}\right)$ as N and M both tend to infinity, where Φ is the integral of the normal distribution

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}dt.$$

Thus, by Lemma 1.3,

$$L(R(N,M)) \leq \Phi\left(\frac{M-N}{\sqrt{N/2}}\right).$$

However, if $W = \{\mathbf{v} \in Z_2^{2N} : |\mathbf{v}|_1 \leq N\}$, then W has 2^{2N-1} elements, yet

$$\text{card}(W + \mathbf{r} \cap W) \leq 2^{2N} \cdot \sqrt{\frac{(N-M)}{M}},$$

for each choice of $\mathbf{r} \in R$, all N, M and $N - M$ large enough.

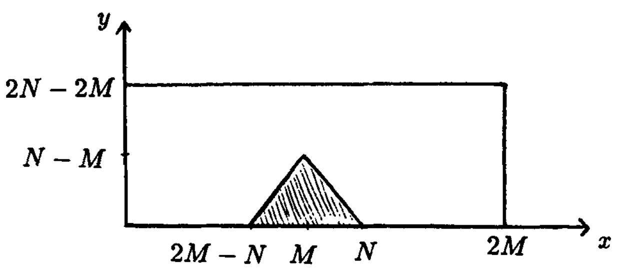

Fig. 2.1.

To see this note that the maximum cardinality is obtained when $|\mathbf{r}|_1 = 2M$ and so, by symmetry, $\mathbf{r}$ indicates the set $\{1, \ldots, 2M\}$, without loss of generality. An element $\mathbf{a}$ of W has s elements inside $\{1, \ldots, 2M\}$ and t elements outside, when considered as a subset of $\{1, \ldots, 2N\}$. Thus $s + t \leq N$. However, if $\mathbf{a} + \mathbf{r}$ is to be in W, then $|\mathbf{a} + \mathbf{r}|_1 = 2M - s + t \leq N$. These conditions on s and t are also sufficient to put $\mathbf{a}$ and $\mathbf{a} + \mathbf{r}$ in W. In this way

$$\text{card}(W + \mathbf{r} \cap W) = \sum_{\substack{s+t \leq N \\ -s+t \leq N-2M \\ s,t \geq 0}} \binom{2M}{s}\binom{2N-2M}{t}.$$

In diagrammatic form, this sum is the sum of the function

$$\Psi(s,t) = \binom{2M}{s}\binom{2N-2M}{t}$$

over the integer pairs to be found in the shaded area of Fig. 2.1.

Provided that N and $N-M$ both tend to infinity, this sum, normalized by dividing by 2^{2N}, therefore becomes asymptotically equal to the integral of the function $\psi'(x,y) = \phi(x)\phi(y), \phi$ being the normal distribution function over the shaded area in Fig. 2.2.

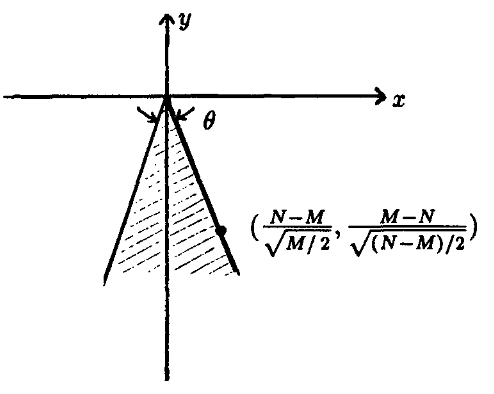

Fig. 2.2.

(See Bollobas [2] for a detailed account of the normal approximation of the binomial distribution.)

Since ψ' is radially symmetric and has integral 1, this area is equal to

$$\frac{\theta}{2\pi} = \frac{1}{\pi}\arctan\sqrt{\frac{N-M}{M}}.$$

So, for N, M and $N-M$ large, card $(W \cap W + \mathbf{r}) \leq 2^{2N} \cdot \sqrt{\frac{N-M}{M}}$, for all $\mathbf{r}$ in $R(N,M)$. Thus $e(1/2; R(N,M)) \leq \sqrt{\frac{N-M}{M}}$, for all N, M and $N-M$ large enough.

In particular, by letting N/M tend to 1 and picking N sufficiently large, one obtains a subset R of Z_2^{2N} for which $L(R)$ is as small as we wish (so that it is a better 'set of recurrence') but for which there is a set W, of density 1/2, for which

$$e = \max\{\text{card } (W \cap W + \mathbf{r})/2^{2N} : \mathbf{r} \in R_n\}$$

is as small as we like. Let R_n be such a sequence of such sets, with associated N_n, M_n, W_n, e_n, etc., for which $L(R_n)$ and e_n both tend to zero.

The following lemma will be important in combining the properties of these sequences into one:

LEMMA 2.2: *Suppose that R and R' are subsets of Z_2^{2N} and $Z_2^{2N'}$ respectively. Let* 0 *be the one element set consisting of the zero vector in either space.*

Let $R'' = R \times 0 \cup 0 \times R'$, a subset of $Z_2^{2N+2N'}$

Then $L(R'') \leq \min\{L(R), L(R')\}$.

Proof: This is clear from the definition of L. ∎

Decompose Z_2^{∞} as a sum $\oplus_{i\geq 1} Z_2^{[b_i, b_{i+1})}$, where $b_{i+1} - b_i = 2N_i$.

Let $R = \cup R'_n$ where $R'_n = 0 \oplus 0 \oplus \cdots \oplus R_n \oplus \cdots, n \geq 1$.

Let $W(n,0) = 0 \oplus 0 \oplus \cdots \oplus W_n \oplus \cdots$ and $W(n,1) = 0 \oplus 0 \oplus \cdots \oplus W_n^c \oplus \cdots$. In these expressions, the non-zero entry appears in the nth place.

Let U be the set $\bigcup_{K,s} \Sigma_{i\in K} W(i, s(i))$, where the union is taken over all finite sets K of integers and functions $s : K \to \{0,1\}$, such that $|s|_1 = \Sigma_{n\in K} s(n)$ is even.

(a) Calculation of the density of U:

$$\begin{aligned} \bar{d} &= \overline{\lim_{n\to\infty}} \frac{|U \cap Z_2^{[1,n]}|}{2^n} \\ &\geq \overline{\lim_{k\to\infty}} \frac{|U \cap Z_2^{[1,b_{k+1})}|}{2^{b_{k+1}-1}} \\ &= \frac{1}{2}. \end{aligned}$$

The last equality derives from the fact that

$$S_k = |\bigcup_{K,s} \sum_{i\in K} W(i, s(i))|,$$

the union being taken over all subsets K of $[1,k]$ and $s : K \to \{0,1\}$, with even ℓ_1 norm, may be rewritten

$$S_k = |\bigcup_{s} \sum_{1\leq i\leq k} W(i, s(i))|,$$

the union now being taken over all $s : [1,k] \to \{0,1\}$ with even ℓ_1 norm.

But each disjoint sum $\Sigma_{1\le i\le k} W(i,s(i))$ has cardinality exceeding $2^{b_{k+1}-k-1}$, and so the union over the 2^{k-1} choices of s yields a value for S_k in excess of $2^{b_{k+1}-2}$. But S_k is equal to $|U \cap Z_2^{[1,b_{k+1})}|$ and so we are done.

(b) The upper density of $U \cap U + \mathbf{r}$:

Suppose that $\mathbf{r}$ is in $R'_k = R \cap Z_2^{[b_k,b_{k+1})}$. By construction, the upper density of $U \cap U + \mathbf{r}$ is bounded above by the number

$$\frac{|W(k,0)\cap(W(k,0)+r)|}{2^{b_{k+1}-b_k}} = \frac{|W(k,1)\cap(W(k,1)+r)|}{2^{b_{k+1}-b_k}} = e_k.$$

Proof of Theorem 2.1: R, constructed above, will do:

It is a set of recurrence, because, if $a > 0$, then there is an n such that $L(R_n) < a$. Therefore $e(a;R) \ge e(a;R_n) > 0$.

However,

$$\overline{\lim_{k\to\infty}} \max\left\{ \lim_{j\to\infty} \frac{|U\cap(U+r)\cap Z_2^{[1,b_{j+1})}|}{2^{b_{j+1}-1}} : r \in R\cap Z_2^{[b_k,b_{k+1})} \right\} = \overline{\lim_{k\to\infty}} e_k = 0$$

and yet

$$\lim_{j\to\infty} \frac{|U\cap Z_2^{[1,b_{j+1})}|}{2^{b_{j+1}-1}} = \frac{1}{2}.$$

So we are done by the observation before. ∎

3.

This section will show how to adapt the example in Z_2^∞ above to give a similar example in $\mathbb{Z}$, and so prove Theorem 1.1. The construction is fairly general, however, and gives rise to an easy exchange of properties between constructions in Z_2^∞ and $\mathbb{Z}$.

First some *definitions* which will make the adaptation easier to describe.

A subset R of Z_2^∞ is said to be **well separated** if there are integers $b_1 < b_2 < \cdots$ and $R_1, R_2, \ldots$, with $Z_2^{[b_i,b_{i+1})} \supseteq R'_i$ and $R = \cup R'_i$, where $R'_i = 0\oplus 0\oplus\cdots\oplus R_i \oplus\cdots$.

Note that the set R constructed in section 2 is well separated.

Let p_i be a sequence of integers, yet to be defined, with the properties: $p_1 = 1$, and $2p_i | p_{i+1}$ for all $i > 0$.

For a given p, let

$$f_p(m) = \begin{cases} 0 & \text{if } 0 \le m < p \bmod 2p, \\ 1 & \text{if } p \le m < 2p \bmod 2p, \end{cases}$$

defined for all $m \in \mathbb{Z}$. Let N be chosen large and let $f : Z_{2p_N} \to Z_2^N$ be defined:

$$f(m) = (f_{p_1}(m), f_{p_2}(m), \dots, f_{p_N}(m)).$$

Let s denote a function $\{1, \dots, N\} \to \{1, -1\}$.

Given s, define $g_s : Z_2^N \to Z_{2p_N}$ by

$$g_s(a_i) = \sum_{1 \le i \le N} s(i) a_i p_i.$$

Given subsets W and R of Z_2^N, define

$$W\hat{} = f^{-1}(W) \qquad \text{and} \qquad R^* = \bigcup_{s:\{1,2,\dots,N\} \to \{-1,1\}} g_s(R)$$

where the union is over all possible choices of sign function s.

Let p_i be a sequence of integers, yet to be determined.

Let $d(W)$ be the density of a subset W of a finite group (there will be no doubt which group this is in the context).

Let R be a subset of this group and define

$$e(W; R) = \max\{d(W \cap W + r) : r \in R\}.$$

LEMMA 3.1: *There is an absolute constant c so that, if W and R are subsets of Z_2^N and N is finite, the following hold:*

(1) $e(W\hat{}; R^*) \le e(W; R) + c \cdot d(W) \sum_{r \in R} \left(w^{2|r|_1} \cdot \sum_{i: r_i = 1} \frac{p_i}{p_{i+1}} \right),$

(2) $L(R^*) \le L(R)$.

Note that this then proves Theorem 1.1:

Proof of Theorem 1.1: By the Furstenberg correspondence (see Furstenberg [4]), it is sufficient to prove that there is a set of recurrence, S in $\mathbb{Z}$, a sequence of intervals $[1, P_n]$ and a set V with the following properties:

$$\lim_{n \to \infty} \frac{|V \cap [1, P_n]|}{P_n} = \frac{1}{2} \text{ and } \lim_{s \in S: s \to \infty} \quad \lim_{n \to \infty} \frac{|V \cap (V + s) \cap [1, P_n]|}{P_n} = 0.$$

It will turn out that suitable choices will be $S = R^*, V = U\hat{}$ and $P_n = 2p_n, p_i$ having been picked sufficiently well.

Note again that the R constructed in section 2 is well separated, i.e. there is a sequence $b_1 < b_2 < \cdots$ of integers, and a sequence of sets R_i such that $Z_2^{[b_i,b_{i+1})} \supseteq R_i$, and $R = \cup R_i'$.

Let $R(n,m) = \bigcup_{m>i\geq n} R_i'$ and $U(n,m) = U \cap Z_2^{[b_n,b_m)}$. Therefore, Lemma 3.1 says that

$$e(U(n,m)\hat{}; R(n,m)^*) \leq e(U(n,m); R(n,m)) + \frac{c}{2} \sum_{r\in R(n,m)} \left(2^{2|r|_1} \cdot \sum_{i:r_i=1} \frac{p_i}{p_{i+1}} \right)$$

and $L(R(n,m)^*) \leq L(R(n,m))$, where the densities etc. are calculated with respect to the group $Z_2^{[b_n,b_m)}$.

If R is a set of recurrence, then $L(R(1,m))$ tends to 0. Thus $L(R^*)$, being dominated by $L(R(1,m)^*)$ and hence by $L(R(1,m))$ for all m, equals zero. So R^* is a set of recurrence.

However, R is also well separated and so the second term in this expression is dominated by

$$\frac{c}{2} \sum_{j\geq n} \left(2^{3|b_{j+1}-b_j|} \cdot \sum_{b_j \leq i < b_{j+1}} \frac{p_i}{p_{i+1}} \right) = \phi_n,$$

which, b_i having been determined, can be made to tend to zero as n tends to infinity, with a careful choice of p_i.

Thus if r^* is in $R(n,n+1)^*$, then $r^* = g_s(r)$ for some $r \in R(n,n+1)$ and sign function s.

$$\begin{aligned} \frac{|U\hat{} \cap (U\hat{} + r^*) \cap [1, 2p_m]|}{2p_m} &\leq d(U(1,m) \cap (U(1,m) + r)) + \phi_n \\ &= d(U(n,n+1) \cap (U(n,n+1) + r)) + \phi_n \end{aligned}$$

which tends to 0 as n tends to infinity by the construction of U in section 2.

Further,

$$\frac{|U\hat{} \cap [1, 2p_m]|}{2p_m} = d(U(1,m)) = \frac{1}{2},$$

by the homogeneity of the ^ construction. So we are done. ∎

Proof of Lemma 3.1: Let $p_i, 1 \leq i \leq N$, be determined and construct R^* and $W\hat{} \in Z_{2p_N}$ from R and $W \in Z_2^N$ as in section 2.

To prove Lemma 3.1 (1) it is sufficient to prove the following inequality which quantifies the degree of the approximation constructed above:

$$\text{(A)} \qquad d((W+r)\hat{} \cap (W\hat{} + g_s(r))) \le c \cdot d(W) \cdot \left(2^{2|r|_1} \cdot \sum_{i:r_i=1} \frac{p_i}{p_{i+1}}\right)$$

for all $r \in Z_2^N, Z_2^N \supseteq W$, and $s : \{1,\dots,N\} \to \{-1,1\}$, where c is an absolute constant.

To complete the proof of Lemma 3.1(1) from here, note that $d(W) = d(W\hat{})$ for all subsets, W, of Z_2^N (the densities are taken over two different, finite, sets, without confusion). Thus

$$e(W;R) \ge d(W \cap W + r) = d((W + r \cap W)\hat{}) = d((W+r)\hat{} \cap W\hat{}), \text{ for all } r \in R.$$

It follows from (A) that, for all choices of $r \in Z_2^N$ and s, a sign function,

$$\begin{aligned} d(W\hat{} \cap (W\hat{} + g_s(r))) &\le e(W;R) + c \cdot d(W) \cdot \left(2^{2|r|_1} \cdot \sum_{i:r_i=1} \frac{p_i}{p_{i+1}}\right) \\ &\le e(W;R) + c \cdot d(W) \sum_{r \in R} \left(2^{2|r|_1} \cdot \sum_{i:r_i=1} \frac{p_i}{p_{i+1}}\right) \end{aligned}$$

for all r and s, and so $e(W\hat{};R^*)$ is dominated by the right hand side of this expression and part (1) of the lemma is proved.

It remains to prove (A). First examine the case $W = \{0\}$:

It is easy to check that, for each $i < N$,

$$|(f^{-1}(0) + p_i)\Delta(f^{-1}(0) - p_i)| = \frac{8p_N p_i}{2^N p_{i+1}}.$$

Thus, if s and t are two sign functions: $\{1,2,\dots,N\} \to \{1,-1\}$, then

$$\text{(B)} \qquad |(f^{-1}(0) + g_s(r))\Delta(f^{-1}(0) + g_t(r))| = 8p_N 2^{-N} \sum_{i \in J} \frac{p_i}{p_{i+1}}$$

where J is the set of indices i for which $r_i = 1$ and $t(i) \neq s(i)$.

For $W = \{w\}$ in general, note that $f^{-1}(w) = f^{-1}(0) + g_1(w)$. Further, if s is a sign function and $r \in Z_2^N$, then

$$g_s(r) + g_1(r) = g_{s'}(r+w) + 2g_1(r') \text{ and } g_t(r) + g_1(r) = g_{t'}(r+w) + 2g_1(r''),$$

where, for example:

$$r_i' = \begin{cases} r_i w_i & \text{if } s(i) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$s'(i) = \begin{cases} s(i) & \text{if } w_i = 0 \text{ and } r_i = 1, \\ 1 & \text{otherwise;} \end{cases}$$

r'' and t' are defined similarly.

Thus

$$\begin{aligned}
&|(f^{-1}(w) + g_s(r))\Delta(f^{-1}(w) + g_t(r))| \\
&\quad \le |(f^{-1}(0) + g_{s'}(r+w) + 2g_1(r'))\Delta(f^{-1}(0) + g_{s'}(r+w))| \\
&\qquad + |(f^{-1}(0) + g_{s'}(r+w))\Delta(f^{-1}(0) + g_{t'}(r+w))| \\
&\qquad + |(f^{-1}(0) + g_{t'}(r+w))\Delta(f^{-1}(0) + g_{t'}(r+w) + 2g_1(r''))| \\
&\quad \le 8p_N 2^{-N} \sum_{i:r_i'=1} \frac{p_i}{p_{i+1}} + 8p_N 2^{-N} \sum_{i:r_i=1} \frac{p_i}{p_{i+1}} + 8p_N 2^{-N} \sum_{i:r_i''=1} \frac{p_i}{p_{i+1}}.
\end{aligned}$$

The first summand comes from the fact that

$$\begin{aligned}
|(f^{-1}(0) &+ g_{s'}(r+w) + 2g_1(r'))\Delta(f^{-1}(0) + g_{s'}(r+w))| \\
&= |(f^{-1}(0) + 2g_1(r'))\Delta(f^{-1}(0))| \\
&= |(f^{-1}(0) + g_1(r'))\Delta(f^{-1}(0) - g_1(r'))| \\
&= 8p_N 2^{-N} \sum_{i:r_i'=1} \frac{p_i}{p_{i+1}}
\end{aligned}$$

from (B) and the third likewise. The middle summand comes from considering the set J of indices on which $r_i + w_i = 1$ and $t'(i) \neq s'(i)$ which, by construction, is contained in the support of r.

Note that the supports of r' and r'' are both contained in the support of r. Therefore,

$$|(f^{-1}(w) + g_s(r))\Delta(f^{-1}(w) + g_t(r))| \le 24 p_N 2^{-N} \sum_{i:r_i=1} \frac{p_i}{p_{i+1}}.$$

For W more general, observe that $W^\wedge = f^{-1}(W)$ is a disjoint union of sets of the form $f^{-1}(w)$, with $w \in W$, and so $(W^\wedge + g_s(r))\Delta(W^\wedge + g_t(r))$ is contained in $\bigcup_{w\in W}((f^{-1}(w) + g_s(r))\Delta(f^{-1}(w) + g_t(r)))$, a set of size less than

$$24 p_N 2^{-N} |W| \sum_{i:r_i=1} (p_i/p_{i+1}).$$

Its density is, therefore, at most $12d(W)\Sigma_{i:r_i=1}(p_i/p_{i=1})$ and (A) is verified and part (1) of the lemma completed.

To prove Lemma 3.1(2) let Z_{2p_N} act on a probability space, $(X, \mathcal{B}, \mu)$, in a measure preserving manner and let A be a measurable subset of X of measure a.

Let $(Y, \mathcal{D}, \nu) = (X, \mathcal{B}, \mu) \times (Z_2^N$, normalized counting measure) define a probability space. Z_2^N acts on Y in a measure preserving manner by permuting the second coordinate:

$$S^{\mathbf{v}}(x, \mathbf{w}) = (x, \mathbf{v} + \mathbf{w}).$$

Let B be the subset of Y defined by $B = \{(x, \mathbf{v}) : x \in T^{g_1(\mathbf{v})}A\}$, a set of measure a.

For every $\varepsilon > 0$, there is an $\mathbf{r}$ in R such that

$$\nu(B \cap S^{\mathbf{r}}B) > e(a; R) - \varepsilon.$$

This implies that there is a $\mathbf{v}$ such that

$$\mu(T^{g_1(\mathbf{v})}A \cap T^{g_1(\mathbf{v}+\mathbf{r})}A) > e(a; R) - \varepsilon.$$

This left hand side equals $\mu(A \cap T^{(g_1(\mathbf{v}+\mathbf{r}) - g_1(\mathbf{v}))}A)$ which equals, in turn, $\mu(A \cap T^{g_s(\mathbf{r})}A)$ where

$$s(i) = \begin{cases} 1 & \text{if } v_i = 0 \text{ or } r_i = 0, \\ -1 & \text{otherwise.} \end{cases}$$

Thus $e(a; R^*) \geq e(a; R) - \varepsilon; \varepsilon$ being arbitrary, we are done.

The whole theorem is now proved. ∎

4. Conclusion

The question of Bergelson dealt with in this paper was well known for some time and it was not obvious which way it would be decided. Indeed, all the previous examples of sets of recurrence displayed very strong recurrence properties, namely that

$$e(a; R \cap [n, \infty)) = a^2 \qquad \text{for all natural numbers } n.$$

This paper shows that this need not happen always and gives a fairly tangible construction of an exception.

The author believes that the above techniques, which were inspired by the paper of Kriz [6], could continue to be quite fruitful in the production of demanding examples in combinatorial ergodic theory.

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